# THE PROBLEM OF PLANE STRAIN OF A PERFECTLY PLASTIC BODY IN TERMS OF COMPLEX VARIABLES

## (ZADACHA O PLOSKOI DEFORMATSII IDEAL'NO PLASTICHESKOGO TELA V KOMPLEKSNYKH PEREMENNYKH)

*PMM Vol.24, No.2, 1960, pp. 367-369*

V. L. DOBROVOL'SKII
(Moscow)

*(Received 4 December 1959)*

This paper presents a general expression for the stress function in terms of complex variables. Certain states of stress are used to illustrate the method. In the case of an elasto-plastic problem the introduction of complex variables permits an investigation of the link between the elastic and the plastic stress functions.

1. For the determination of the state of stress in a plastic region for plane strain of a perfectly plastic body we make use of the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1.1}$$

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2 \tag{1.2}$$

and the boundary conditions

$$[\sigma_x \cos(nx) + \tau_{xy} \cos(ny)]_\gamma = X_n(s), \ [\tau_{xy} \cos(nx) + \sigma_y \cos(ny)]_\gamma = Y_n(s) \tag{1.3}$$

where $\gamma$ denotes the boundary of the plastic region and $s$ is the arc length along the contour $\gamma$.

We will introduce the stress function $F_1$ by the formulas

$$\sigma_x = \frac{\partial^2 F_1}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F_1}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F_1}{\partial x\, \partial y} \tag{1.4}$$

Equations (1.1) will then be satisfied indentically, and for the determination of the function $F_1$ there remains Equation (1.2). In changing to complex variables, we note that (1.2) may be written in the form

$$M\overline{M} = 4k^2, \qquad M = \sigma_y - \sigma_x + 2i\tau_{xy}$$

On the other hand, one has

$$M = 4\frac{\partial^2 F_1}{\partial z^2}, \qquad \overline{M} = 4\frac{\partial^2 \overline{F}_1}{\partial \overline{z}^2}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Thus, (1.2) takes on the form

$$4\frac{\partial^2 F_1}{\partial z^2}\frac{\partial^2 \overline{F}_1}{\partial \overline{z}^2} = k^2 \tag{1.5}$$

Introducing the function $F$ by the formula

$$F = \frac{2}{k}F_1$$

we obtain the equation for the determination of the stress function in plastic region

$$\frac{\partial^2 F}{\partial z^2}\frac{\partial^2 \overline{F}}{\partial \overline{z}^2} = 1 \tag{1.6}$$

The boundary conditions (1.3) now reduce to [1]

$$\left[\frac{\partial F}{\partial \overline{z}}\right]_\gamma = \frac{2}{k}f(s),\ f(s) = i\int_{s_0}^{s}(X_n + iY_n)\,ds + \text{const} \tag{1.7}$$

It follows from (1.6) that

$$\frac{\partial^2 F}{\partial z^2} = \exp(2i\theta) \tag{1.8}$$

where $\theta = \theta(z, \overline{z})$ is an arbitrary real function. Integrating (1.8), we obtain the general solution of (1.6)

$$F(z, \overline{z}) = \int_{z_0}^{z} d\eta \int_{\eta_1}^{\eta} \exp[2i\theta(\xi, \overline{z})]\,d\xi + \overline{z\varphi(z)} + \overline{\chi(z)} \tag{1.9}$$

where $\phi(z)$ and $\chi(z)$ are arbitrary analytic functions. For the solution to have physical meaning, i.e. for it to correspond to some state of stress, one must impose the condition that the function $F$ is real:

$$\int_{z_0}^{z} d\eta \int_{\eta_0}^{\eta} \exp[2i\theta(\xi, \overline{z})]\,d\xi + z\overline{\varphi(z)} + \overline{\chi(z)} = \int_{\overline{z}_0}^{\overline{z}} d\overline{\eta} \int_{\overline{\eta}_0}^{\overline{\eta}} \exp[-2i\theta(z, \overline{\xi})]\,d\overline{\xi} + \overline{z}\varphi(z) + \chi(z) \tag{1.10}$$

For given functions $\theta(z, \overline{z})$ the solution (1.9) is determined by the condition (1.10) exactly, within a term $pz\overline{z}$ (where $p$ is real) which corresponds to the presence of hydrostatic pressure. For the solution of concrete boundary-value problems one has to impose on the solution (1.9), in order that the requirement (1.10) be fulfilled, the boundary condition in the form

$$\left[\int_{\bar{z}_0}^{\bar{z}} \exp\left[-2i\theta(z,\bar{\xi})\right] d\bar{\xi} + \varphi(z)\right]_{\gamma} = \frac{2}{k} f(s) \tag{1.11}$$

We note that between the angle of inclination $\alpha$ of a slip line to the $x$-axis and the function $\theta(z, \bar{z})$, at every point of the region of plasticity, there exists the relationship

$$\theta + \alpha = \frac{1}{4}\pi \tag{1.12}$$

**2.** Consider next the state of stress for specific functions $\theta(z, \bar{z})$.

1) For $\theta = \alpha$ ($\alpha$ = const) we obtain

$$F = \frac{1}{2}\exp(2i\alpha) z^2 + \frac{1}{2}\exp(-2i\alpha)\bar{z}^2 + pz\bar{z} \tag{2.1}$$

$$\frac{\sigma_x}{k} = p - \cos 2\alpha, \qquad \frac{\sigma_y}{k} = p + \cos 2\alpha, \qquad \frac{\tau_{xy}}{k} = \sin 2\alpha \tag{2.2}$$

This solution corresponds to a homogeneous stress field.

2) For $\theta = -\vartheta + \alpha$ ($\alpha$ = const, $z = re^{i\vartheta}$)

$$F = \exp(2i\alpha)\bar{z}z(\ln z - 1) + \exp(-2i\alpha)z\bar{z}(\ln\bar{z} - 1) + pz\bar{z} \tag{2.3}$$

Here the following versions are possible:

*a*) Axisymmetric stress distribution [2] ($\alpha$ = 0)

$$\frac{\sigma_r}{k} = \ln r^2 - 1 + p, \qquad \frac{\sigma_\vartheta}{k} = \ln r^2 - 1 + p, \qquad \frac{\tau_{r\vartheta}}{k} = 0 \tag{2.4}$$

*b*) Stress distribution [2] in a wedge ($\alpha$ = 1/4 $\pi$)

$$\frac{\sigma_r}{k} = -2\vartheta + p, \qquad \frac{\sigma_\vartheta}{k} = -2\vartheta + p, \qquad \frac{\tau_{r\vartheta}}{k} = 1 \tag{2.5}$$

*c*) Superposition of the preceding stress fields [3]

$$\begin{aligned}\frac{\sigma_r}{k} &= \cos 2\alpha(\ln r^2 - 1) - 2\vartheta\sin 2\alpha + p, \\ \frac{\sigma_\vartheta}{k} &= \cos 2\alpha(\ln r^2 + 1) - 2\vartheta\sin 2\alpha + p,\end{aligned} \qquad \frac{\tau_{r\vartheta}}{k} = \sin 2\alpha \tag{2.6}$$

3) The general case of axisymmetric stress distribution [4]. For

$$\theta = \frac{1}{2}\tan^{-1}\frac{\varepsilon c}{\sqrt{r^4 - c^2}} - \vartheta \qquad (\varepsilon = \pm 1,\ c = \text{const})$$

we obtain

$$F = \varepsilon\left[z\bar{z}\ln\left(z\bar{z} + \sqrt{(z\bar{z})^2 - c^2}\right) - 2\sqrt{(z\bar{z})^2 - c^2} - c\sin^{-1}\frac{c}{z\bar{z}}\right] - ic\ln z + ic\ln\bar{z} + pz\bar{z} \tag{2.7}$$

For this

$$\frac{\sigma_r}{k} = \varepsilon\left[2\ln(\sqrt{r^2-c}+\sqrt{r^2+c}) - \frac{\sqrt{r^4-c^2}}{r^2}\right] + p,\qquad \frac{\tau_{r\vartheta}}{k} = \frac{c}{r^2} \qquad (2.8)$$
$$\frac{\sigma_\vartheta}{k} = \varepsilon\left[2\ln(\sqrt{r^2-c}+\sqrt{r^2+c}) + \frac{\sqrt{r^4-c^2}}{r^2}\right] + p,$$

4) For $\theta = 1/2 \sin^{-1}(z-\bar{z})/i$

$$F = \frac{1}{6}\sqrt{[1+(z-\bar{z})^2]^3} - \frac{1}{2}\sqrt{1+(z-\bar{z})^2} - \frac{z-\bar{z}}{2i}\tan^{-1}\frac{z-\bar{z}}{i\sqrt{1+(z-\bar{z})^2}} +$$
$$+\frac{1}{6}(z^3+\bar{z}^3) - \frac{1}{2}(z\bar{z}^2+\bar{z}^2 z) + pz\bar{z}$$

$$\frac{\sigma_x}{k} = -2x + 2\sqrt{1-4y^2} + p,\qquad \frac{\sigma_y}{k} = -2x + p \qquad \frac{\tau_{xy}}{k} = 2y \qquad (2.9)$$

Such a stress distribution will occur in a strip $(-1/2 < y < +1/2)$, compressed by rough plates [3].

5) For $\theta = -\vartheta + 1/2 \sin^{-1}(1 - c/r^2)(c > 0)$

$$F = i(c+z\bar{z})\ln\frac{z}{\bar{z}} - 3\sqrt{2cz\bar{z}-c^2} + 2(c+z\bar{z})\tan^{-1}\frac{\sqrt{2cz\bar{z}-c^2}}{c} + pz\bar{z} \qquad (2.10)$$

$$\frac{\sigma_r}{k} = -2\vartheta - \frac{1}{r^2}\sqrt{2cr^2-c^2} + 2\tan^{-1}\frac{\sqrt{2cr^2-c^2}}{c} + p,\qquad \frac{\tau_{r\vartheta}}{k} = 1 - \frac{c}{r^2} \qquad (2.11)$$
$$\frac{\sigma_\vartheta}{k} = -2\vartheta + \frac{1}{r^2}\sqrt{2cr^2-c^2} + 2\tan^{-1}\frac{\sqrt{2cr^2-c^2}}{c} + p,$$

This is a new particular solution of the equilibrium equations (1.1) and the von Mises condition (1.2).

3. In the case of elasto-plastic problems we have for the determination of the stress functions the following equations:

In the elastic region

$$\frac{\partial^4 F^\circ}{\partial z^2\,\partial\bar{z}^2} = 0 \qquad (3.1)$$

In the plastic region (1.8) or

$$\frac{\partial^4 F}{\partial z^2\,\partial\bar{z}^2} = \frac{\partial^2}{\partial\bar{z}^2}\exp[2i\theta(z,\bar{z})] \qquad (3.2)$$

On the dividing lines between the elastic and plastic zones one has to use the condition of continuity of the first derivatives of $\overset{\circ}{F}$ and $F$. Since the stress functions $F$ must be real, it follows from (1.9) that it may be represented in the form

$$F(z, \bar{z}) = F_0(z, \bar{z}) + \varkappa(z, \bar{z}) \tag{3.3}$$

where

$$F_0(z, \bar{z}) = 2\,\mathrm{Re}[\bar{z}\varphi(z) + \chi(z)], \qquad \varkappa(z, \bar{z}) = \int_{z_0}^{z} d\eta \int_{\eta_0}^{\eta} \exp[2i\theta(\xi, \bar{z})]\, d\xi - \bar{z}\varphi(z) - \chi(z)$$

and $\kappa(z, \bar{z})$ is a real function. The function $\kappa(z, \bar{z})$ must vanish at the interface and in the elastic region. The condition of continuity of the derivatives of the stress functions on the boundary may be satisfied by letting

$$\left[\frac{\partial F_0}{\partial \bar{z}}\right]_\gamma = \left[\frac{\partial F^\circ}{\partial \bar{z}}\right]_\gamma, \qquad \left[\frac{\partial \varkappa}{\partial \bar{z}}\right]_\gamma = 0 \tag{3.4}$$

For this fulfilment of the boundary conditions the function $\kappa(z, \bar{z})$ and its first derivatives with respect to $z$, $\bar{z}$ at the interface and in the elastic region vanish, i.e. the solution is continued analytically from the elastic into the plastic region.

## BIBLIOGRAPHY


1. Muskhelishvili, N.I., Nekotorye osnovnye zadachi matematicheskoi teorii uprugosti (Some basic problems of the mathematical theory of elasticity). *Izd-vo Akad. Nauk SSSR* 1954.

2. Galin, L.A., Ploskaia uprugo-plasticheskaia zadacha (The plane elasto-plastic problem). *PMM* Vol. 10, No. 3, 1946.

3. Sokolovskii, V.V., Teoriia plastichnosti (Theory of plasticity). *Izd-vo Akad. Nauk SSSR* 1946.

4. Mikhlin, S.G., Osnovnye uravneniia matematicheskoi teorii plastichnosti (The basic equations of the mathematical theory of plasticity). *Izd-vo Akad. Nauk SSSR* 1934.


*Translated by* J.R.M.R.